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Rook problem on a fractal lattice

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Abstract. The number of rook arrangements on the Sierpinski lattice is presented as a set of recurrence formulae. The upper and lower bounds of this number are also found.

1. Introduction

Rook problem [4–8] is one of the enumerative problems of counting the number of different arrangements of rooks on a lattice under the restriction that rooks cannot attack each other. The most famous eight-Queens (or N-Queens) problem has been attracting interest for many years, but its solution has not been found [1, 2]. Here we present a solution of the rook problem on the Sierpinski lattice (see figure 1) which is a fractal lattice [3]. Some fractal lattices can be constructed by a recursive operation from a primary lattice. For example, Dragon curve, Koch's snowflake curve, etc [3]. We denote the Sierpinski lattices by $E(n)$, where n denotes the number of times the recursive operation is applied. A primary lattice is denoted by $E(1)$ in figure 1.

Our rook-arrangements obey the following three rules.

(i) Each rook is on a different site which is an intersection of line segments.

(ii) If a rook is on a site of a line, it can go to any site of the line by one move.

(iii) No rook can attack any others by one move.

Then, our problem is *'How many arrangements of rooks are there on $E(n)$ under the above rules?'*, and we solve it for the case of the Sierpinski lattice.

Throughout this paper, the number of arrangements is denoted by $C(n)$ corresponding to the lattice $E(n)$. We show four arrangements of rooks in figure 1, where a full circle on a site indicates a rook. The first three examples are allowed under our rules, but the fourth one is not allowed.

2. On the Sierpinski lattice

Theorem 1. The rook arrangements-number $C(n)$ on the Sierpinski lattice $E(n)$ is given by

$$C(n) = C_0(n) + 3C_1(n) + 3C_2(n) + C_3(n) = 4\widetilde{C_0}(n) + 6\widetilde{C_1}(n) + 3\widetilde{C_2}(n) + \widetilde{C_3}(n) \qquad (1)$$

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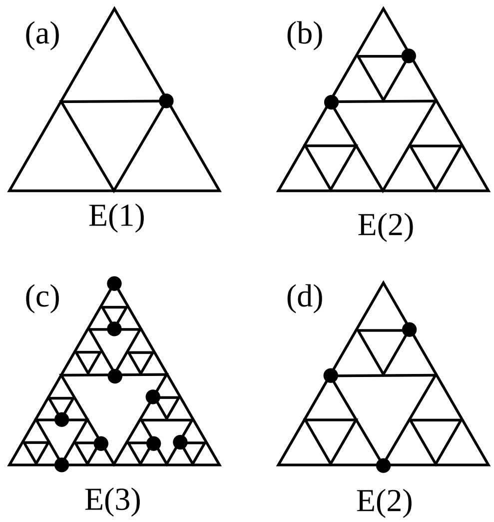


Figure 1. Examples of the rook-arrangement on the Sierpinski lattices $E(1)$, $E(2)$, $E(3)$ and $E(2)$. The full circles are rooks. The first three configurations (a)–(c) are allowed, but the last one (d) is forbidden.

where $\widetilde{C}_0(n), \ldots, \widetilde{C}_3(n)$ obeys the following recurrence equations

$$\begin{aligned}
\widetilde{C}_0(n+1) &= \widetilde{C}_0^{\,3}(n) + 3\widetilde{C}_0^{\,2}(n)\widetilde{C}_1(n) + 3\widetilde{C}_0(n)\widetilde{C}_1^{\,2}(n) + \widetilde{C}_1^{\,3}(n) \\
\widetilde{C}_1(n+1) &= \widetilde{C}_0^{\,3}(n) + 3\widetilde{C}_0^{\,2}(n)\widetilde{C}_1(n) + 4\widetilde{C}_0(n)\widetilde{C}_1^{\,2}(n) + 2\widetilde{C}_1^{\,3}(n) \\
&\quad + 2\widetilde{C}_0^{\,2}(n)\widetilde{C}_2(n) + 4\widetilde{C}_0(n)\widetilde{C}_1(n)\widetilde{C}_2(n) + 2\widetilde{C}_1^{\,2}(n)\widetilde{C}_2(n) \\
\widetilde{C}_2(n+1) &= 4\widetilde{C}_0^{\,2}(n)\widetilde{C}_1(n) + 5\widetilde{C}_0(n)\widetilde{C}_1^{\,2}(n) + 3\widetilde{C}_1^{\,3}(n) + 3\widetilde{C}_0^{\,2}(n)\widetilde{C}_2(n) \\
&\quad + 8\widetilde{C}_0(n)\widetilde{C}_1(n)\widetilde{C}_2(n) + 7\widetilde{C}_1^{\,2}(n)\widetilde{C}_2(n) + 3\widetilde{C}_0(n)\widetilde{C}_2^{\,2}(n) \\
&\quad + 3\widetilde{C}_1(n)\widetilde{C}_2^{\,2}(n) + \widetilde{C}_0^{\,2}(n)\widetilde{C}_3(n) + 2\widetilde{C}_0(n)\widetilde{C}_1(n)\widetilde{C}_3(n) \\
&\quad + \widetilde{C}_1^{\,2}(n)\widetilde{C}_3(n) \\
\widetilde{C}_3(n+1) &= 9\widetilde{C}_0(n)\widetilde{C}_1^{\,2}(n) + 5\widetilde{C}_1^{\,3}(n) + 3\widetilde{C}_0^{\,2}(n)\widetilde{C}_2(n) + 12\widetilde{C}_0(n)\widetilde{C}_1(n)\widetilde{C}_2(n) \\
&\quad + 12\widetilde{C}_1^{\,2}(n)\widetilde{C}_2(n) + 6\widetilde{C}_0(n)\widetilde{C}_2^{\,2}(n) + 12\widetilde{C}_1(n)\widetilde{C}_2^{\,2}(n) + 2\widetilde{C}_2^{\,3}(n) \\
&\quad + 3\widetilde{C}_0^{\,2}(n)\widetilde{C}_3(n) + 6\widetilde{C}_0(n)\widetilde{C}_1(n)\widetilde{C}_3(n) + 6\widetilde{C}_1^{\,2}(n)\widetilde{C}_3(n) \\
&\quad + 6\widetilde{C}_0(n)\widetilde{C}_2(n)\widetilde{C}_3(n) + 6\widetilde{C}_1(n)\widetilde{C}_2(n)\widetilde{C}_3(n)
\end{aligned} \tag{2}$$

and the initial condition of these recursive equations is given by

$$\widetilde{C}_0(1) = \widetilde{C}_1(1) = 1 \qquad \widetilde{C}_2(1) = \widetilde{C}_3(1) = 0. \tag{3}$$

Symbols $C_i(n)$ and $\widetilde{C}_i(n)$, $(i = 0, 1, 2, 3)$ denote the number of the rook-arrangements for the corresponding modified lattices introduced in the proof of the theorem.

Before going to the proof, we check the simplest case for the primary lattice $E(1)$. As shown in figure 2, there is one arrangement with no rook (we should count the no rook case every time). There are six different arrangements with one rook, since $E(1)$ has six sites. Two rooks can be put on $E(1)$ by three different configurations. We cannot put three or more rooks on this lattice under our rule. Consequently the number $C(1)$ is

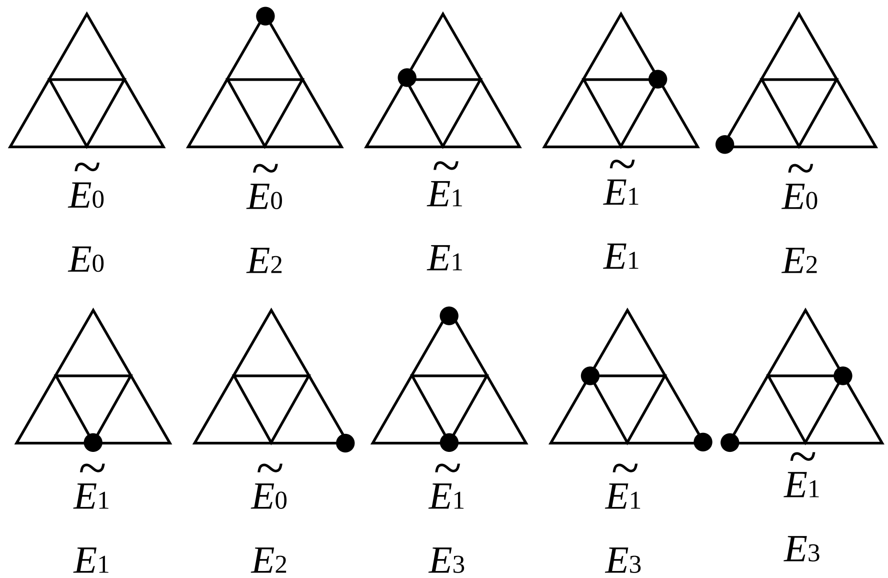


Figure 2. Ten rook-arrangements on the primary Sierpinski lattice $E(1)$. One no-rook case, six one-rook cases and three two-rooks cases. The symbols under each figure indicate which type each arrangement belongs to. The symbols are explained in the proof of theorem 1.

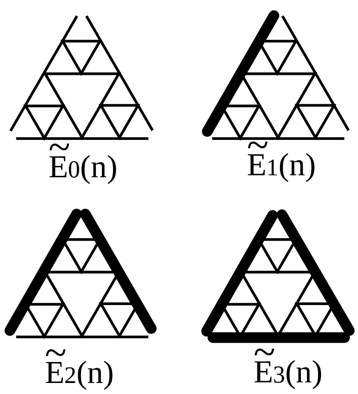


Figure 3. $\widetilde{E}_i(n)$ for $i = 0, 1, 2, 3$. The outer bold line has a rook on it.

$1 + 6 + 3 = 10$. This result agrees with that of theorem 1. From equations (1) and (3), we have $C(1) = 4 \times 1 + 6 \times 1 + 3 \times 0 + 0 = 10$.

Proof of theorem 1. We introduce a modified lattice by removing three vertices of the largest triangle of the original lattice $E(n)$, as shown in figure 3. We call this modified lattice $\widetilde{E}(n)$. We use a tilde to indicate this modified lattice.

The lattice $\widetilde{E}(n)$ is classified into four types depending on how the rooks are put on its three outer lines as follows.

$\widetilde{E_0}$: No rook is on any site of outer three lines.

$\widetilde{E_1}$: One rook is on a site of one outer line and other two lines have no rook.

$\widetilde{E_2}$: Two rooks are on an each site of two outer lines respectively. The third outer line has no rook.

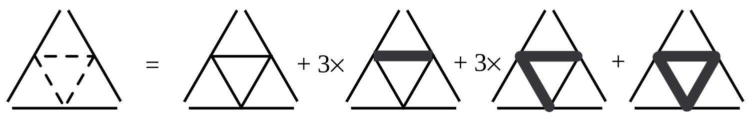
$\widetilde{E_3}$: Three rooks are on an each site of three outer lines respectively.

There is no other type of lattice under the rook condition such that each outer line has at most one rook. Thus we have $\widetilde{E} = \widetilde{E_0} + 3\widetilde{E_1} + 3\widetilde{E_2} + \widetilde{E_3}$, here the factor 3 for $\widetilde{E_1}$ and $\widetilde{E_2}$ comes from the symmetry of their lattices. The corresponding rook-arrangements number is denoted by $\widetilde{C_i}(n)$ for $\widetilde{E_i}(n)$ with $i = 0, 1, 2, 3$.

Proof of equation (2). First, we show that $\widetilde{C_0}(n+1)$ is given by

$$\widetilde{C_0}(n+1) = \widetilde{C_0}^3(n) + 3\widetilde{C_0}^2(n)\widetilde{C_1}(n) + 3\widetilde{C_0}(n)\widetilde{C_1}^2(n) + \widetilde{C_1}^3(n). \tag{4}$$

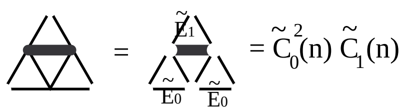
The reason is shown graphically below. $\widetilde{E_0}(n+1)$ is constructed by three $\widetilde{E}(n)$,



(5)

Here the plain line means that no rook is on any site of this line, the bold line means that a rook is on a some site of this line and the broken line means either of the two cases.

The second term on the right-hand side of equation (5) is constructed by two $\widetilde{E_0}(n)$ and one $\widetilde{E_1}(n)$ as

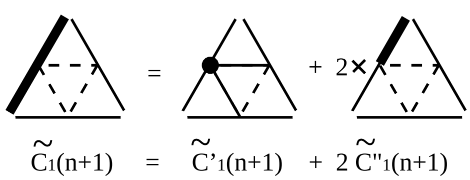


. (6)

Here we should take care to ensure that $\widetilde{E_1}(n)$ cannot be replaced with a rotated one. We can place a bottom-bold $\widetilde{E_1}(n)$ at the top triangle of $\widetilde{E_0}(n+1)$, or a left-bold one at the right, or a right-bold one at the left. These three cases are taken into account by the factor 3 for the second term in equation (5).

Other terms of equation (4) are estimated similarly. The first term of equation (5) is constructed by three $\widetilde{E_0}$, the third term is by one $\widetilde{E_0}$ and two $\widetilde{E_1}$, the fourth term is by three $\widetilde{E_1}$. Then we obtain equation (4).

Next we consider $\widetilde{C_1}(n+1)$. Let us divide $\widetilde{C_1}(n+1)$ into two cases $\widetilde{C_1}'(n+1)$ and $\widetilde{C_1}''(n+1)$, the former has a rook on a centre site of the left outer line of $\widetilde{E_1}(n+1)$ and the latter has a rook on the remaining some site of the left outer line. Thus we find



. (7)

Similarly in equations (5) and (6), the number $\widetilde{C_1}'(n+1)$ is given by

$$\widetilde{C_1}'(n+1) = \widetilde{C_0}^3(n) + \widetilde{C_0}^2(n)\widetilde{C_1}(n) \tag{8}$$

and the number $\widetilde{C_1}''(n+1)$ is given by

$$\begin{aligned}\widetilde{C_1}''(n+1) = {} & \widetilde{C_0}^2(n)\widetilde{C_1}(n) + 2\widetilde{C_0}(n)\widetilde{C_1}^2(n) + \widetilde{C_1}^3(n) + \widetilde{C_0}^2(n)\widetilde{C_2}(n) \\ & + 2\widetilde{C_0}(n)\widetilde{C_1}(n)\widetilde{C_2}(n) + \widetilde{C_1}^2(n)\widetilde{C_2}(n).\end{aligned} \tag{9}$$

With (7)–(9), we find the second equation in (2).

The numbers $\widetilde{C}_2(n+1)$ and $\widetilde{C}_3(n+1)$ can be derived by using similar arguments to the above. Then, we have the set of recurrence equations (2) in theorem 1. □

Proof of equation (3). As shown in figure 2, the number for the primary lattices $\widetilde{E}_i(1)$ denoted by $\widetilde{C}_i(1)$, $i = 0, 1, 2, 3$ are given by

$$\widetilde{C}_0(1) = \widetilde{C}_1(1) = 1 \qquad \widetilde{C}_2(1) = \widetilde{C}_3(1) = 0 \tag{10}$$

since the $\widetilde{E}(1)$ lattice only has three sites ($\widetilde{E}(1)$ does not have outer three vertices) and this lattice has at most one rook. This is the initial condition of the recurrence equations (3) in theorem 1. □

Proof of equation (1). To return to the original lattice $E(n)$, we restore three vertices which were removed at the beginning of the proof to $\widetilde{E}(n)$ lattice. The original lattice is also classified into four types as similarly as $\widetilde{E}(n)$ lattice.

E_0: No rook is on any site of outer three lines.

E_1: One rook is on any site of one outer line and other two lines have no rook.

E_2: Two outer lines are occupied by one or two rooks. One outer line is not.

E_3: All three outer lines are occupied.

We set $C_i(n)$ be the number of rook-arrangements of $E_i(n)$ lattice, where $i = 0, 1, 2, 3$.

E_0 and E_1 are defined to be the same as $\widetilde{E}_0$ and $\widetilde{E}_1$, respectively, but E_2 and E_3 are different. This difference comes from whether the added vertices are occupied by a rook or not. The relation between E_i and $\widetilde{E}_i$ are shown in figure 4. E_0 and E_1 have no rook on the outer vertices. A rook on an added outer vertex occupies two outer lines of $E(n)$ and then this type of the configuration belongs to $E_2(n)$ or $E_3(n)$. Only one rook can be put on one of the three vertices. $E_2(n)$ has one rook on a vertex or has two rooks on the two outer lines. The number of the former case is $\widetilde{C}_0(n)$ and the number of the latter case is $\widetilde{C}_2(n)$. $E_3(n)$ has one rook on a vertex and one rook on the line opposite to this vertex ($3\widetilde{C}_1(n)$) or it has three rooks on three outer lines ($\widetilde{C}_3(n)$). Then we have

$$\begin{aligned} &C_0(n) = \widetilde{C}_0(n) \qquad C_1(n) = \widetilde{C}_1(n) \\ &C_2(n) = \widetilde{C}_0(n) + \widetilde{C}_2(n) \qquad C_3(n) = 3\widetilde{C}_1(n) + \widetilde{C}_3(n). \end{aligned} \tag{11}$$

Finally, taking account of the symmetry of the configuration, we have the number of the rook-arrangement $C(n)$ by

$$C(n) = C_0(n) + 3C_1(n) + 3C_2(n) + C_3(n). \tag{12}$$

With equations (11) and (12), we arrived at equation (1) of theorem 1. □

The numbers $C(n)$ for $n = 1, 2, \ldots, 7$ are given in table 1.

3. Upper and lower bounds

We have an upper bound and a lower bound of $C(n)$.

Theorem 2. The number of the rook arrangements is bounded as

$$10^{3^n} > C(n) \geqslant 60^{2^{\frac{n-1}{2}}}/6 \tag{13}$$

where n is assumed to be odd.

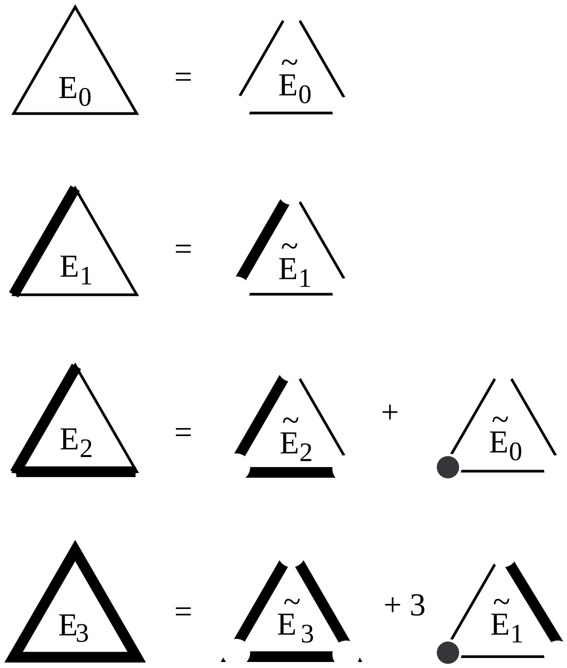


Figure 4. The relation between $\widetilde{E}_i$ and E_i for $i = 0, 1, 2, 3$.

Table 1. The number of rook arrangements $C(n)$ for small n.

n	$C(n)$
1	10
2	142
3	340568
4	2582700948625408
5	35634397855122371766195392380102272297310617 6
6	1905353318182952468259652553125739730162277424356212171492677368211762 7336474444859199889941609510226859141124824224813142769664000
7	4612663800978825897597481066193343647160936051992368347266393888999005 7614196370380224152346528099661255070309544062430172984361069480567193 5933682124305560247111933579692120775011517869773314297619464237103929 3708464147088719594619409394203939113543717368099359942064435941712709 2003304387267516560537371224651156925919221629088900480063886298360384 0034621732021175255918622801920000000000000

Proof of theorem 2. It is easily seen from figure 5 that $C(n) < C^3(n-1)$, because three $E(n-1)$ overlap each other. Some rook-configurations are forbidden in $E(n)$ but are allowed in three independent $E(n-1)$ configurations. By solving this recurrence inequality equation, we have $C(n) < 10^{3^n}$.

Since we can place two $E(n-2)$ without any overlapping we obtain a lower bound of $C(n)$ in the same manner that $C(n) \geqslant 6C^2(n-2)$. The factor of 6 comes from the number

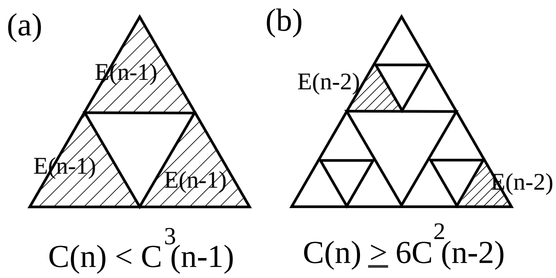


Figure 5. (*a*) Upper and (*b*) lower bounds. There is no rook in the areas without shade.

of ways of placing two $E(n-2)$ without overlapping. Consequently, we found that $C(n)$ grows double exponentially as n increases. □

By applying the least square method to the data of $C(1), \ldots, C(7)$ in table 1, we estimate $C(n)$ as

$$\log\log C(n) \simeq 1.054n - 0.5172. \tag{14}$$

It is interesting that the upper bound in equation (13) is a good approximate value for equation (14), i.e.

$$\log\log 10^{3^n} = n\log 3 + \log\log 10 \simeq 1.0986n + 0.8340. \tag{15}$$

How fast does the value $C(n)$ grows in comparison with other lattices? We consider a square chess board whose size is $2^n \times 2^n$. This lattice is a special case of fractal lattice. The number of rook-arrangements $C(n)$ of this lattice is given by

$$C_c(n) = \sum_{k=0}^{2^n} \frac{(2^n!)^2}{(2^n-k)!(k!)^2}. \tag{16}$$

It is obvious that

$$\frac{2^{n+1}!}{(2^n!)^2} = \sum_{k=0}^{2^n} \left\{ \frac{(2^n!)}{(2^n-k)!k!} \right\}^2 < C_c(n) < \sum_{k=0}^{2^n} \frac{(2^n!)^2}{(2^n-k)!} = 2^{2^n} 2^n!. \tag{17}$$

Thus we have the leading term as

$$\log\log C_c(n) \simeq n\log 2 \simeq 0.6931n. \tag{18}$$

Comparing equations (14) and (18), we conclude that the value $C(n)$ of the Sierpinski lattice grows faster than that of the square chess board.

4. Summary and discussion

This paper provides a set of recurrence equations for the rook-arrangements number $C(n)$ for the Sierpinski lattice. We also find the upper and the lower bounds of it. From these bounds, we proved that this number $C(n)$ grows double exponentially as n increases. The obtained recurrence equations (2) have not been solved analytically yet.

There are remaining problems. How many rook arrangements are there when we cannot put any more rooks at any sites? This should be called the fourth rule. An example allowed under the four rules is shown in figure 6. We cannot add any rook to this configuration. The original N-Queen problem obeys these four rules.

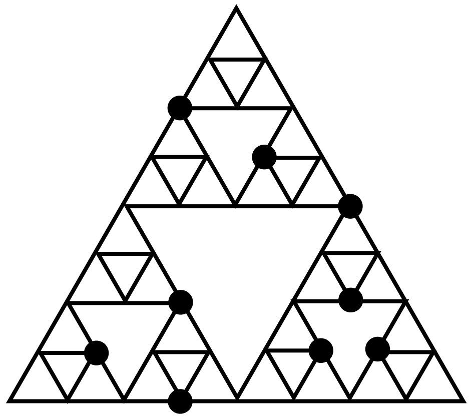


Figure 6. An example rook configurations for $E(3)$. No more rooks can be put on any site.

Recently, Chalub *et al* [9] used a similar recursive method to the self-avoiding-walk problem on an extended Sierpinski lattice and discussed the limit of the Euclidean lattice. Their method is an interesting approach to solving the original N-Queen problem.

Acknowledgment

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